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Fooled by Robustness: A Radius of Stability Perspective

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Abstract

One would have expected the considerable public debate created by Nassim Taleb's two best selling books on uncertainty, *Fooled by Randomness* and *The Black Swan*, to inspire a greater sensitivity to the fundamental difficulties posed by *severe uncertainty*. Yet, surprisingly, this is not always the case. So, the objective of this discussion is to call attention to an example of an incautious approach to severe uncertainty that is manifested in the proposition to use the concept *radius of stability* as a measure of robustness against severe uncertainty. We examine a prime exponent of this approach to decision-making under severe uncertainty, namely *info-gap decision theory,* whose central proposition is precisely this: use a simple radius of stability model to quantify, analyze and manage severe uncertainty. This discussion is a reminder that the generic radius of stability model is a model of *local* robustness. It is therefore utterly unsuitable for the treatment of severe uncertainty when the latter is characterized by a poor estimate of the parameter of interest, a vast uncertainty space, and a likelihood-free quantification of uncertainty.

Key words: Robustness, radius of stability, local, severe uncertainty, info-gap decision theory, maximin

1 INTRODUCTION

This discussion was motivated by the recent publication in this and other peer-reviewed journals, of a significant number of articles advocating the use of *radius of stability* models for the quantification, analysis and management of *severe* uncertainty.

We find this fact surprising because:

- One would have expected that the issues raised in Nassim Taleb's [1,2] two popular books, *Fooled by Randomness* and *The Black Swans: The Impact of the Highly Improbable*, would have prompted risk analysis scholars to take a more reflective, more cautious stance towards uncertainty.
- Considering that the generic radius of stability model [3–10] is a model that is designed to measure robustness against *small perturbations* in a given nominal value of a parameter, it is patently unsuitable for the modeling, analysis and management of situations where the perturbations of interest can be *extremely large.*
- The rich literature on robust decision-making in the face of severe uncertainty, which is the product of tremendous progress over the past twenty years in the area of *robust optimization*, offers an array of models for the quantification of *global* robustness [11–14].

We note in passing that it would certainly be interesting, indeed instructive, to examine why radius of stability models are misapplied to situations where models of *global* robustness are required. However, this question is not on our agenda.

Rather, our main objective here is to make it clear that the concept *radius of stability* serves to measure *local* robustness. Hence, methodologically and practically, not only is it utterly unsuitable for the quantification, modeling, analysis and management of *severe* uncertainty, it is in fact the exact anti-thesis of what a model for the treatment of severe uncertainty ought to be.

The discussion is thus organized as follows:

- Section 2: *Radius of stability.* A brief discussion on the generic radius of stability model.
- Section 3: *Local vs global robustness.* A reminder of the difference between local and global robustness.
- Section 4: *Robustness against severe uncertainty.* A reminder of the meaning of *severe uncertainty* and a brief explanation of the reasons rendering radius of stability models unsuitable for the management of such uncertainty.
- Section 5: *Info-gap decision theory.* A brief discussion of info-gap's robustness model.
- Section 6: *A robust optimization perspective.* A broad overview of *robust optimization* and the role of Wald's Maximin model in this area.
- Section 7: *Examples.* Two examples illustrating the difference between local and global robustness.
- Section 8: *Info-gap decision theory revisited.* An examination of info-gap decision theory from two perspectives with the view to determine its role and place in decision theory and robust optimization.
- Section 9: *Where is the limit?* A discussion on the consequences of the blurring of the distinction between local and global robustness.
- Section 10: *Robust optimization revisited.* A speculative attempt to answer the question: where is Robust Optimization ?!?!?
- Section 11: *Conclusions.*

2 RADIUS OF STABILITY

There are many situations where our objective is to analyze the impact of small perturbations in the nominal value of a parameter on the stability of the system under consideration. So, it was precisely with this object in mind that the *radius of stability* model was formulated in the 1960s [3,4]. Today this model is used extensively in diverse fields such as applied mathematics, numerical analysis, control theory, operations research, optimization theory, economics, and so on [5–10].

Described in broad terms, the radius of stability of an object (parameter, matrix, function) is the size of the largest perturbation (in the worst direction) in the nominal value of the object that does not destabilize it. That is, it is a measures of how much the object can deviate (uniformly) from its nominal value without being destabilized.

Since in general the parameter of interest can be a multi-dimensional object, say a vector, a matrix, or even a function, the radius of stability of an object is a measure of the object's robustness to the "worst-case perturbation". That is, the radius of stability of an object is the size of the largest perturbation (in the 'worst' direction) in the nominal value of the object that does not destabilize it. And this is precisely what makes this concept a measure of (local) *robustness*.

From a purely mathematical point of view, this is an extremely convenient measure of (local) robustness, because, it reduces a *multi-dimensional* perturbation to a *real number*.

To describe this intuitive concept formally, consider a system $q \in Q$ whose *state* $s \in S(q)$ can be either *stable* or *unstable* depending on whether it satisfies a given *stability requirement*. That is, system q is stable iff $s \in \widehat{S}(q)$, where $\widehat{S}(q)$ is the subset of $S(q)$ comprising the *stable* states in $S(q)$. We shall refer to $S(q)$ as the *state space* of system q and to $\widehat{S}(q)$ as the *region of stability* of system q.

The radius of stability model addresses then the following intuitive and practical question: *How robust is system q to small perturbations in a given nominal value of the state?*

To define this concept formally, let $\tilde{s}$ denote the nominal value of the state, and let $B(\rho, \tilde{s})$ denote a ball of radius ρ centered at $\tilde{s}$ based on some measure of *distance* between states in $S(q)$. That is, $B(\rho, \tilde{s})$ is the subset of $S(q)$ comprising the states that are within distance ρ from $\tilde{s}$. For simplicity we assume that the nominal state $\tilde{s}$ is stable, namely that $\tilde{s} \in \widehat{S}(q)$. Then,

Radius of stability of $q \in Q$ at $\tilde{s} \in \widehat{S}(q)$:

$$\hat{\rho}(q, \tilde{s}) := \max_{\rho \geq 0} \{\rho : s \in \widehat{S}(q), \forall s \in B(\rho, \tilde{s})\} \tag{1}$$

In words:

> The radius of stability of a system at a given nominal state is the radius of the *largest ball* centered at the nominal state, all of whose elements are stable. Equivalently, it is the size of the *smallest perturbation* in the nominal state that *destabilizes* the system. The larger the radius of stability, the more stable (robust) the system is to small perturbations in the nominal state $\tilde{s}$.

This intuitive concept is illustrated in Figure 1, where the rectangle represents $S(q)$ and the shaded area represents the region of stability $\widehat{S}(q)$.

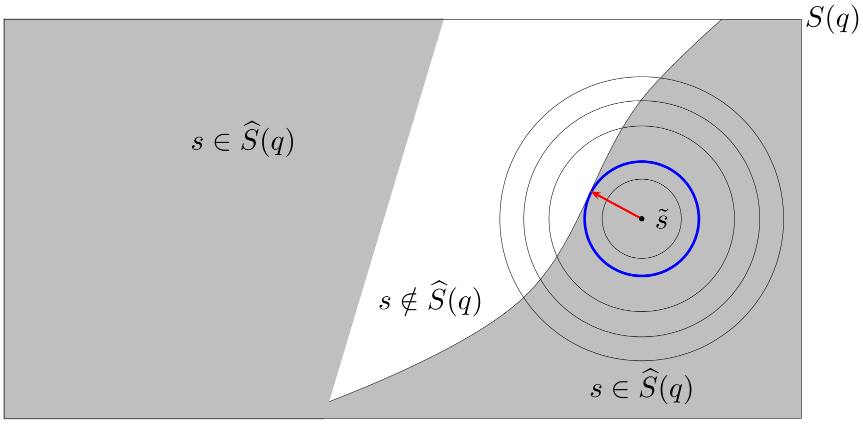

Figure 1: Radius of stability of system q at $\tilde{s}$

Much as this picture brings out that, methodologically, the radius of stability does not provide a measure of *global* stability, simply because it does not measure stability with respect to variations in the value of s over the entire state space $S(q)$ of system q, to amplify the point let us consider the situation shown in Figure 2.

Note that, here, relatively small perturbations in $\tilde{s}$ can destabilize system q', yet the system is stable in large areas of the state space $S(q') = S(q)$ that are distant from $\tilde{s}$.

So, the question that one would no doubt ask is this: *Why then should we be concerned about large perturbations in $\tilde{s}$ when even small perturbations can destabilize it?*

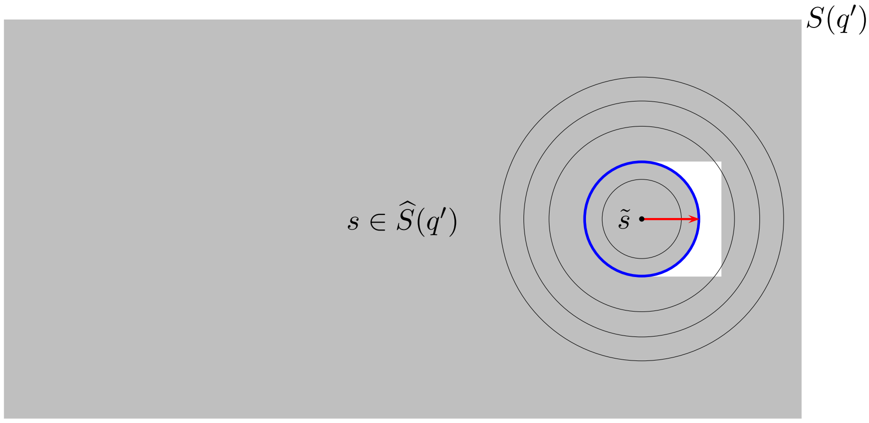

Figure 2: Radius of stability of system q' at $\tilde{s}$, with $S(q) = S(q')$

This, of course, is a thoroughly legitimate question and we shall address it, albeit briefly, in the next section.

3 LOCAL vs GLOBAL ROBUSTNESS

Having established that the radius of stability model is a typical model of *local* robustness, designed to handle situations where the objective is to determine a system's robustness to changes in the parameter of interest in the neighborhood of a given value of the parameter, the following question arises.

What about situations where robustness is sought over *global* variations in the parameter of interest? For instance, in the case of the system depicted in Figure 2, we might want to determine the robustness of system q' to global changes in the value of $\tilde{s}$ over $S(q')$. In this case we would ask: How stable is system q' over its state space $S(q')$?

Clearly, the radius of stability of system q does not provide an indication of the stability of q over its entire state space $S(q)$.

So, consider the following intuitive *global* measure of stability [11, 15]:

$$\gamma(q) := \frac{size(\widehat{S}(q))}{size(S(q))} \tag{2}$$

where $size(A)$ denotes the *size* of set A. We shall not go into all the minutia detailing the requirements that $size(A)$ must meet to count as the "size" of set A, except to require that $size(A)$ be non-negative and that $A \subseteq B$ entails that $size(A) \leq size(B)$.

In words, $\gamma(q)$ is the fraction of the total number of states of system q that are *stable.* In broad terms, $\gamma(q) = 0$ means that q is thoroughly unstable, whereas $\gamma(q) = 1$ means that system q is very stable, in fact, super stable: all its states are stable. Similarly, $\gamma(q) = 0.25$ means that 25% of the states of q are stable.

For example, in the context of the systems shown in Figure 1 and Figure 2, if we let $size(A)$ denote the *area* associated with set A, and assuming that the areas shown in this figures are indicated by the same units, $\tilde{s} = \tilde{s}'$ and $S(q) = S(q') = S$, then clearly $\gamma(q')$ is much larger than $\gamma(q)$, hence q' is more stable (globally) than q on S. It is important to take note that the statements:

- q' is more globally-robust than q on S.
- q' is less locally-robust than q at $\tilde{s}$.

are not contradictory. *Local* and *global* robustness are related, but quite different, concepts.

It is therefore instructive to bring out the difference between local and global robustness by considering the following two abstract situations:

Local Robustness	Global Robustness
The system is currently in state $s = \tilde{s}$. We know that the state will gradually change, but we have no clue in what 'direction'. So, we ask: what is the largest deviation (in the worst 'direction') from the current state that will not destabilize the system?	We know that the future state of the system will be some $s \in S(q)$ but we have no clue which one it might be. So, we ask: how robust is the system to the uncertainty in its future state?

These, one need hardly point out, are two totally different situations that call for two totally different treatments.

Furthermore, what should be gathered from this short discussion on *global* robustness is this. Because the 'radius of stability', by definition, addresses situations of local robustness, it is *not* a measure of global robustness and should therefore not be used for this purpose, unless, of course, it can be shown, in the case of a problem being considered, that it can provide a suitable measure of global robustness.

In many respects, the similarities and differences between local and global robustness are analogous to the similarities and differences between local and global *optimum.*

In the next section we discuss situations that call for the use of a *global* measure of robustness where the radius of stability model is manifestly the wrong means for this purpose.

4 ROBUSTNESS AGAINST SEVERE UNCERTAINTY

Consider the case where s represents a parameter whose *true value* is unknown, furthermore is subject to *severe uncertainty,* $S(q)$ represents the set of possible values of the true value of s, and $\tilde{s}$ represents an *estimate* of the true value of s. Also assume that the severity of the uncertainty is manifested in these features:

- The uncertainty space $S(q)$ is *vast*, can even be *unbounded.*
- The estimate $\tilde{s}$ is very *poor,* can be a *guess,* even a *wild guess.*
- The uncertainty over $S(q)$ is *likelihood-free.*

The third feature entails that there is no ground to assume that the true value of s is more/less likely to be in any one particular neighborhood of $S(q)$. Specifically, that there is no ground to assume that the true value of s is more/less likely to be in the neighborhood of the estimate $\tilde{s}$ than in the neighborhood of any other state in $S(q)$.

Then, this being the case, and especially in view of the severity of the uncertainty in the true value of s, it stands to reason that we would want to identify values of s not only in the immediate neighborhood of the poor estimate $\tilde{s}$, but also values of s that are at a large distance from $\tilde{s}$.

And to see why, assume that $S(q)$ in Figure 1 is equal to $S(q')$ in Figure 2 and that both systems have the same estimate and the same radius of stability.

Then, even if the uncertainty is severe, we can clearly argue that system q' is more robust than system q against the uncertainty in the true value of s because the region of stability of q', namely $\widehat{S}(q')$, is much larger than the region of stability of q, namely $\widehat{S}(q)$.

To illustrate, suppose that the true value of s is *uniformly distributed* on the state space. Then the *probability* that the system is stable is equal to the ratio of the area of the region of stability and the area of the state space. By inspection, this probability is much larger for system q' than for system q. Hence, if robustness is measured by the *probability* that the state is stable, then system q' is more (globally) robust than q' against the severe uncertainty in the true value of s.

All this leads to the inevitable conclusion that there is no rational/justification for using a model of *local* robustness, such as the radius of stability model, to quantify, analyze and manage *severe uncertainty.*

And yet, the proposition to take this approach to severe uncertainty is made by a theory that claims to be at the vanguard of decision-making under severe uncertainty, namely *info-gap decision theory* [16–18].

5 INFO-GAP DECISION THEORY

The origins of this theory go back to the study of the reliability of mechanical systems [19] where it was used as a framework for the evaluation of the robustness of systems against perturbations in the nominal value of a parameter of interest. The robustness model deployed for this purpose was a simple radius of stability model, where the stability requirements are typically specified by inequalities of the form $r^* \leq r(q,s)$, or $r^* \geq r(q,s)$ where r^* is a numerical scalar representing a critical performance level and $r(q,s)$ represents the performance level of system q when it is in state s. It ought to be pointed out, though, that despite this, no awareness is shown in this book of the fact that the robustness model used is a radius of stability model.

Then, in 2001 the very same robustness theory was relaunched as a *decision theory,* the crucial point of departure being that robustness was now treated as a criterion for the *ranking* of decisions [16]. The larger the robustness, the better the decision, hence the best (optimal) decision is that whose robustness is the largest.

Formally then,

Info-gap's Robustness Model:

$$\hat{\rho}(q,\tilde{s}) := \max_{\rho \geq 0} \left\{\rho : r^* \leq r(q,s), \forall s \in B(\rho,\tilde{s})\right\}, \; q \in Q \tag{3}$$

Info-gap's Decision Model:

$$\hat{\rho}(\tilde{s}) := \max_{q \in Q} \; \hat{\rho}(q,\tilde{s}) \tag{4}$$

$$= \max_{q \in Q} \max_{\rho \geq 0} \left\{\rho : r^* \leq r(q,s), \forall s \in B(\rho,\tilde{s})\right\} \tag{5}$$

$$= \max_{q \in Q, \rho \geq 0} \left\{\rho : r^* \leq r(q,s), \forall s \in B(\rho,\tilde{s})\right\} \tag{6}$$

In words,

> The robustness of decision q is the size (radius) of the largest ball around the estimate $\tilde{s}$ such that the performance requirement $r^* \leq r(q,s)$ is satisfied at all the values of s in that ball.
> The optimal decision is a decision whose robustness is the largest.

Formally then, from a radius of stability perspective:

> Info-gap's robustness model is a radius of stability model characterized by the property that the region of stability is specified as follows:

$$\widehat{S}(q) = \left\{s \in S(q) : r^* \leq r(q,s)\right\}, q \in Q \tag{7}$$

But what is really important for our discussion is the fact that, in its 2001 incarnation, info-gap decision theory was relaunched not just as a decision theory, but as a decision theory that furnishes a *new* methodology for the treatment of *severe uncertainty*.

Info-gap decision theory's prowess as a methodology for severe uncertainty was depicted in claims such as this [16, p. 208] (emphasis is added):

> **Most** of the commonly encountered info-gap models are **unbounded.**

In the second edition of the book [17], the characterization of info-gap decision theory's great prowess at handling *severe uncertainty* was further amplified by the numerous references to "Knightian uncertainty", namely non-probabilistic, likelihood-free uncertainty, and in the claims that info-gap's methodology is particularly well-placed to manage situations subject to these conditions.

What we do not find in these books is any reference to the radius of stability model nor any allusion to the fact that the robustness model deployed by the theory is a model of **local robustness**. Instead, we find claims that the theory is new and radically different [17, p. xii] (emphasis is added):

> Info-gap decision theory is **radically different** from **all** current theories of decision under uncertainty. The difference originates in the modelling of uncertainty as an information gap rather than as a probability. The need for info-gap modeling and management of uncertainty arises in dealing with severe lack of information and highly unstructured uncertainty.

and [17, p. 11]

In this book we concentrate on the fairly **new** concept of information-gap uncertainty, whose differences from more classical approaches to uncertainty are **real and deep.** Despite the power of classical decision theories, in many areas such as engineering, economics, management, medicine and public policy, a need has arisen for a different format for decisions based on severely uncertain evidence.

And in the new book on info-gap decision theory we read [18, p. x] (emphasis is added):

The management of **surprises** is central to the "economic problem", and info-gap theory is a response to this challenge.

Having said all that, the following questions are inevitable:

- How can info-gap decision theory possibly be described as "fairly new" and "radically different" when its robustness model is clearly a simple instance of the radius of stability model (circa 1960) — the most prevalent model of local robustness?
- How can info-gap decision theory possibly be able to handle "unbounded" uncertainty spaces when it is based on a definition of *local* robustness?
- How can it possibly deal with "surprises" when its robustness analysis is conducted in the neighborhood of a poor estimate?

These questions are taken up in a comprehensive critique of *info-gap decision theory*, where the critique is conducted primarily from a decision theory point of view [10]. In what follows we briefly address these questions from the viewpoint of *robust optimization*.

6 A ROBUST OPTIMIZATION PERSPECTIVE

The roots of robust optimization go back to modern decision theory [20, 21] (founded in the 1950s), notably to Wald's Maximin model [22–24] which straightaway established itself as the foremost non-probabilistic, worst-case-analysis type, paradigm for decision-making in the face of severe uncertainty.

The emergence of robust optimization as a separate field in optimization theory, mathematical programming and operations research, dates back to the early 1970 [11–14, 25, 26].

Broadly speaking, a robust optimization problem is an optimization problem involving a parameter with respect to which robustness is sought. That is, a solution to a robust optimization problem is a solution that "performs well" with respect to the optimization task under consideration, for a range of values of the parameter. For example, the following is a typical (abstract) robust optimization problem:

$$\max_{x\in X} \ g(x) \text{ subject to } h(x,u) \geq c \ , \ \forall u \in \mathscr{U} \tag{8}$$

where X and $\mathscr{U}$ are some sets, g is a real valued function on X, h is a real valued function on $X \times \mathscr{U}$, and c is a given numeric scalar.

The quest for robustness is this case is entailed in the clause $\forall u \in \mathscr{U}$, namely in the requirement that for $x \in X$ to be *admissible,* the constraint $h(x,u) \geq c$ must be satisfied by *all* values of the parameter u in $\mathscr{U}$.

It is clear that a connection to decision-making under *severe uncertainty* is immediate: the set $\mathscr{U}$ would represent the set of all possible values of a parameter u whose "true" value is subject to severe uncertainty. The severity of the uncertainty would be manifested in the fact that the quantification of the uncertainty is *likelihood-free*. Namely, the assumption would be that all we know about the true value of u is that it is an element of $\mathscr{U}$. No likelihood structure (e.g. probability distribution) would be attributed to the uncertainty space $\mathscr{U}$.

It should be stressed, though, that the field of robust optimization is not concerned only with problems for which we lack the information and data to formulate a likelihood-based (e.g. probabilistic) uncertainty model. In fact, the field of robust optimization deals with probabilistic models of uncertainty as well. But it seems the most obvious discipline to turn to when we would seek to obtain a likelihood-free formulation of a problem considered.

The prominent position that robust optimization holds today in such areas as operations research and mathematical programming is due to progress over the past twenty years in two related but distinct facets of robust optimization, namely

· Models of robustness.

· Algorithms for generic robust optimization problems.

That the link between these two facets is vital is obvious, because it is one thing to formulate a "nice" robust optimization model that is most suitable for the analysis of a given problem, but often, solving the concrete robust optimization problem specified by this model is quite another. For instance, some of the attractive robustness models that were developed in the early days of robust optimization [11, 12, 15] give rise to extremely difficult optimization problems that can be solved in practice only if certain simplifying assumptions hold [27–31].

The "art" in robust optimization is then to find a proper balance between a mathematical model that is suitable for the problem under consideration and an algorithm that is capable of solving the optimization problem specified by the model.

With this as background, we note that of particular interest to us in this discussion is the following class of (constrained) robust optimization problems:

$$\beta(q) := \max_{x \in X} \; g(x) \;\; \text{subject to } r(q,u) \geq r^*, \forall u \in U(x) \tag{9}$$

where q represent an external object (e.g. system, alternative), r^* is a numeric scalar, r is a real valued function on $Q \times \mathscr{U}$ and for each $x \in X$ the set $U(x)$ is a subset of some set $\mathscr{U}$.

Also note that for the special case where $g(x) \equiv x$ and $X = [0, \infty)$, this class of problems takes the following format

$$\beta(q) := \max_{x \geq 0} \; \{x : r(q,u) \geq r^*, \forall u \in U(x)\} \tag{10}$$

Thus, if $U(x)$ is a ball of radius x centered at some point $\tilde{u}$ such that $U(0) = \{\tilde{u}\}$, then this model is an *info-gap robustness* model, hence also a radius of stability model, in disguise.

In short, from the perspective of robust optimization, info-gap's robustness model is a very simple robust optimization model where the robustness is characterized by the requirement $r(q,u) \geq r^*, \forall u \in U(x)$ and the sets $U(x), x \geq 0$, are balls of radius x centered at the same point $\tilde{u}$.

But, the point we hasten to make is that much as these models are simple robust optimization models, they are models of *local* robustness and are therefore the **wrong** models for the treatment of *severe* uncertainty that is characterized by vast uncertainty spaces. Indeed, the use of such models to this end represents a "somewhat irresponsible decision maker" [32, p. 926]. That is, it represents a decision maker who ignores the performance of decisions in areas of the uncertainty space that are outside the "normal range" of values of the parameter of interest. Sniedovich [10] calls these areas *No-Man's Land* and the prescription for decision-making under severe uncertainty that is based on such models is deemed "Voodoo" decision-making.

To illustrate, consider the case where the uncertainty space is unbounded, say $\mathscr{U} = (-\infty, \infty)$, as shown in Figure 3. The value of the robustness of system/decision q is completely independent of the performance of q, as measured by $r(q,u)$, over the *No Man's Land* region

$$NML(q) := \mathscr{U} \setminus [\tilde{u} - \beta(q) - \varepsilon, \tilde{u} + \beta(q) + \varepsilon] \tag{11}$$

$$= (-\infty, \tilde{u} - \beta(q) - \varepsilon) \bigcup (\tilde{u} + \beta(q) + \varepsilon, \infty) \tag{12}$$

for some arbitrarily small $\varepsilon > 0$.

The point is that, in general, there is no ground to assume that the performance of a system on a minute subset of a vast uncertainty space is representative of its performance on the entire uncertainty space. Thus, a system whose local robustness at $\tilde{u}$ is high/low is not necessarily robust/fragile against the severe uncertainty in the true value of u over $\mathscr{U}$.

In Section 7 we illustrate this point through a concrete example where the most robust decision in the neighborhood of the nominal value of the parameter is not the most robust decisions over the entire uncertainty space $\mathscr{U}$.

6.1 Wald's maximin model

One of the most important paradigms for the treatment of severe uncertainty in robust optimization is Wald's Maximin model [22–24]. For the purposes of this discussion it is convenient to consider the

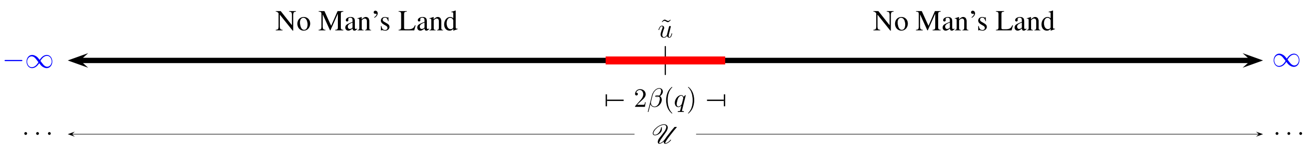

Figure 3: The *No Man's Land* property of radius of stability models

following two equivalent formats of this model, namely the *classic* format and the *mathematical programming* (MP) format:

$$\begin{array}{ccc} & \text{Wald's Maximin model} & \\ \text{Classic format} & & \text{MP format} \\ \max_{d\in D}\ \min_{\sigma\in\Sigma(d)} f(d,\sigma) & \equiv & \max_{\substack{d\in D\\ z\in\mathbb{R}}} \{z : z \leq f(d,\sigma), \forall \sigma \in \Sigma(d)\} \end{array} \tag{13}$$

where D denotes the *decision space,* $\Sigma(d)$ denotes the set of *states* associated with decision d and $f(d,\sigma)$ denotes the *payoff* to the decision maker if she selects decision d and state σ is realized.

Constraints can be incorporated explicitly in these formats. For example, the following equivalent formats incorporate the constraint $g(q,\sigma) \geq 0$ in the formulation of the problem, where q is a given external object:

Constrained Classic format :

$$\max_{d\in D}\ \min_{\sigma\in\Sigma(d)} \{f(d,\sigma) : g(q,\sigma) \geq 0, \forall \sigma \in \Sigma(d)\} \tag{14}$$

Constrained MP format :

$$\max_{\substack{d\in D\\ z\in\mathbb{R}}} \{z : z \leq f(d,\sigma), g(q,\sigma) \geq 0, \forall \sigma \in \Sigma(d)\} \tag{15}$$

Now, consider the instance of the (constrained) Maximin model that is associated with the radius of stability model via the following specification:

Maximin model		Radius of stability model
d	$\equiv$	ρ
D	$\equiv$	$[0,\infty)$
σ	$\equiv$	s
$\Sigma(d)$	$\equiv$	$B(\rho,\tilde{s})$
$f(d,s)$	$\equiv$	ρ
$g(q,\sigma) \geq 0$	$\equiv$	$s \in \widehat{S}(q)$

and

$$g(q,\sigma) = \begin{cases} 0 & , \ \sigma \in \widehat{S}(q) \\ -1 & , \ \sigma \notin \widehat{S}(q) \end{cases} \tag{16}$$

This instance of the generic Maximin model yields the following simple Maximin model:

$$\overbrace{\max_{\substack{d\in D\\ z\in\mathbb{R}}} \{z : z \leq f(d,\sigma), g(q,\sigma) \geq 0, \forall \sigma \in \Sigma(d)\}}^{\text{Constrained Maximin model (MP format)}} \tag{17}$$

$$\equiv \max_{\substack{\rho\geq 0\\ z\in\mathbb{R}}} \{z : z \leq f(\rho,s), g(q,s) \geq 0, \forall s \in B(\rho,\tilde{s})\} \tag{18}$$

$$\equiv \max_{\rho\geq 0} \{\rho : \rho \leq f(\rho,s), s \in \widehat{S}(q), \forall s \in B(\rho,\tilde{s})\} \tag{19}$$

$$\equiv \max_{\rho\geq 0} \{\rho : s \in \widehat{S}(q), \forall s \in B(\rho,\tilde{s})\} \tag{20}$$

$$\equiv \underbrace{\max \{\rho \geq 0 : s \in \widehat{S}(q), \forall s \in B(\rho,\tilde{s})\}}_{\text{radius of stability model}} \tag{21}$$

We therefore conclude that the radius of stability model is an instance of Wald's Maximin model. This implies in turn that info-gap's robustness model is equally an instance of Wald's Maximin model [10,33–35].

For the record, we point out that the intuitive global measure of robustness based the "size criterion" (2) can be formulated as a Maximin model as follows:

$$z^*(q) := \max_{S \subseteq S(q)} \min_{s \in S} h(S, s, q) \tag{22}$$

where

$$h(S, s, q) := \begin{cases} \gamma(q, S) & , \ s \in \hat{S}(q) \\ -1 & , \ s \notin \hat{S}(q) \end{cases} \tag{23}$$

$$\gamma(q, S) := \frac{size(S)}{size(S(q))} \ , \ S \subseteq S(q) \tag{24}$$

observing that the -1 is a penalty used to ensure that the decision (set S) selected by the $\max$ player does not contain any unacceptable (unstable) values of s. This yields the equivalent MP format:

$$z^*(q) := \max_{S \subseteq S(q)} \min_{s \in S} h(S, s, q) \equiv \max_{S \subseteq S(q)} \{\gamma(q, S) : s \in \widehat{S}(q), \forall s \in S\} \tag{25}$$

We hasten to add, though, that these intuitive models of global robustness are rarely used in practice. The difficulty is that, short of certain simplifying conditions being satisfied, the optimization problems induced by them are practically intractable [15,27–31].

6.2 Global robust optimization models

To illustrate a global approach to robustness, consider the following uncertainty-free *parametric* optimization problem:

Problem P(s):

$$\max_{q \in Q} f(q; s) \ \text{ subject to } \ r^* \leq r(q; s) \tag{26}$$

where f and r are real-valued functions on Q and $s \in S$ is a given parameter (the informal "; s" notation is a reminder that s is a *parameter,* not an explicit *argument,* of the function).

Now consider a similar problem, except that here the value of s is not given, rather it is subject to severe uncertainty. Let $S(q)$ denote the set of possible values of s associated with decision $q \in Q$. Given these conditions, it stands to reason that we would attempt to solve this problem via the *robust counter-part* of Problem P(s), namely

Problem R:

$$\max_{q \in Q} \min_{s \in S(q)} \{f(q, s) : r^* \leq r(q, s), \forall s \in S(q)\} \tag{27}$$

where in this framework f and r have two arguments, namely q and s. Observe that in this framework, s is an explicit argument of f and r and that it is the *decision variable* of the minimizer.

The difficulty is, however, that typically there is no decision $q \in Q$ such that

$$r^* \leq r(q, s) \ , \ \forall s \in S(q) \tag{28}$$

hence, Problem R may not have feasible solutions, let alone optimal solutions.

One possible way to resolve this difficulty is to relax (28) and require instead

$$r^* \leq r(q, s) \ , \ \forall s \in \mathcal{N} \tag{29}$$

where $\mathcal{N} \subset S(q)$ represents the "normal range" of the state s.

States outside this set would be admissible only under "controlled" violations of the constraint, in the sense that the further s is from $\mathcal{N}$, the greater its license to violate the performance constraint $r^* \leq r(q, s)$.

Following this line, we can adopt the *globalized* approach [14, 32, 36] to robustness to "relax" the global constraint (28) as follows:

$$r^* \leq r(q,s) + \beta \cdot dist(s,\mathcal{N}) \ , \ \forall s \in S(q) \tag{30}$$

where $\beta \geq 0$ is a control parameter and $dist(s,\mathcal{N})$ denotes the distance from s to $\mathcal{N}$ based on some suitable metric, such that $dist(s,\mathcal{N}) \geq 0, \forall s \in S(q)$ and $dist(s,\mathcal{N}) = 0, \forall s \in \mathcal{N}$. Note that the relaxed constraint entails (29) and for $\beta = 0$, it reverts to the strict constraint (28).

We can then incorporate this relaxed global constraint in a Maximin model as follows:

Problem G:
$$\max_{q\in Q} \min_{s\in S(q)} \{f(q,s) : r^* \leq r(q,s) + \beta \cdot dist(s,\mathcal{N}), \forall s \in S(q)\} \tag{31}$$

In practice, the "normal range" $\mathcal{N}$ can be parameterized by its "size" which can be varied in a *sensitivity analysis* framework.

Note the difference between the three sets of forbidden (unacceptable) $(s, r(q,s))$ values for decision q:

$$\mathcal{F}_{strict}(q) := \{(s,y) : s \in S(q), y < r^*\} \tag{32}$$
$$\mathcal{F}_{relaxed}(q) := \{(s,y) : s \in S(q), y < r^* - \beta \cdot dist(s,\mathcal{N})\} \tag{33}$$
$$\mathcal{F}_{normal}(q) := \{(s,y) : s \in \mathcal{N}, y < r^*\} \tag{34}$$

In the next section we illustrate this idea and highlight the difference between local and global robustness.

7 EXAMPLES

For the sake of simplicity, the two examples below are presented graphically rather than algebraically. The first features a simple radius of stability analysis, and the second a global robustness analysis that is anchored in the discussion in the preceding section.

7.1 Radius of stability analysis

Consider the case where $Q = \{q', q'', q''')\}$, namely where there are three decisions and where the uncertainty spaces are all equal to the real line $\mathbb{R}$: that is, $S(q') = S(q'') = S(q''') = (-\infty, \infty)$. Also, assume that the respective estimates are all equal to 0, namely $\tilde{s}' = \tilde{s}'' = \tilde{s}''' = 0$.

We consider the case where, in accordance with the precepts of info-gap decision theory, the stability regions are determined by a performance requirement $r^* \leq r(q,s)$. The critical performance level r^* is equal to 10, and the performance functions are shown in Figure 4. Since the uncertainty space is unbounded, only a small section of it, in the neighborhood of the estimate, is shown. Assume that the performance functions continue their trends, as shown in the figure, in both directions.

For simplicity, suppose that the balls used are of the form $B(\rho, \tilde{s}) = [s : |s - \tilde{s}| \leq \rho]$, in which case we have $B(\rho, 0) = [-\rho, \rho], \rho \geq 0$.

Since decision q' violates the performance requirement at the estimate $\tilde{s} = 0$, it follows that its radius of stability is equal to zero: $\rho' = 0$. By inspection, the radii of stability of q'' and q''' are $\rho'' = 4$ and $\rho''' = 5$, respectively.

Hence, according to the radius of stability approach, the most robust decision at $s = 0$ is q'''. Observe, however, that this decision is not at all robust over its uncertainty space $S(q''') = (-\infty, \infty)$.

In contrast, although according to the precepts of the radius of stability model decision q'' is not as robust as q''' at $s = 0$, it is far more robust than q''' globally on the uncertainty space $(-\infty, \infty)$.

But this, one need hardly point out, is hardly surprising because, as we have seen above, the radius of stability is not designed to handle global robustness.

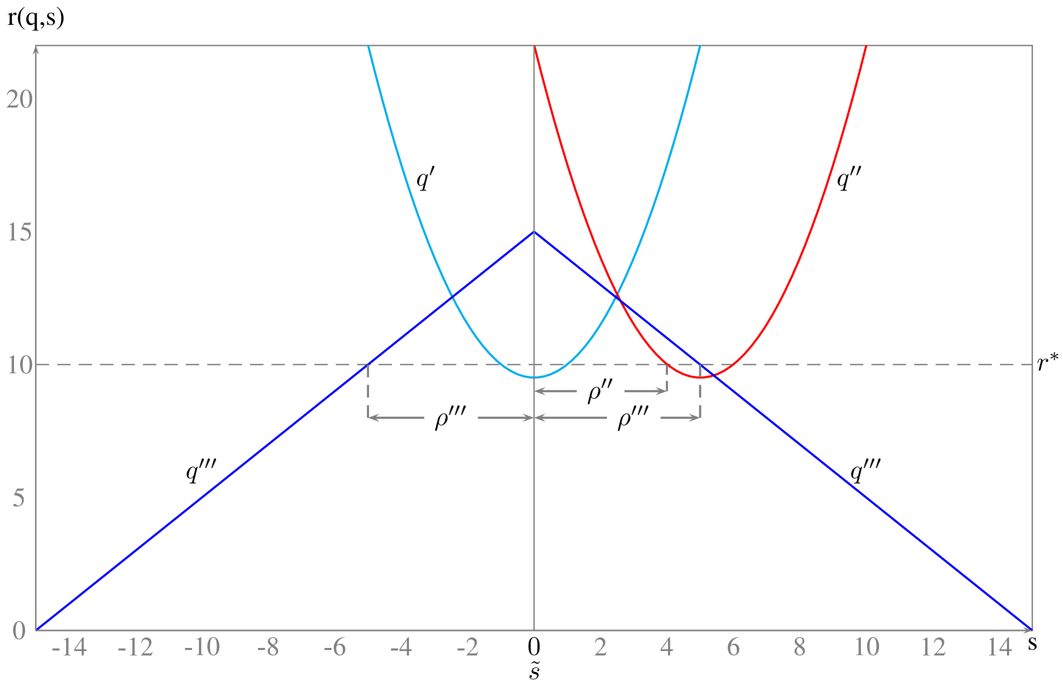

Figure 4: Radius of stability analysis

7.2 A global robustness model

Consider the instance of Problem G specified as follows:

$$Q = \{q', q'', q''', q''''\}$$
$$S(q') = S(q'') = S(q''') = S(q'''') = (-\infty, \infty)$$
$$\tilde{s}' = \tilde{s}'' = \tilde{s}''' = \tilde{s}'''' = 0$$
$$r^* = 10$$
$$\mathcal{N} = [-2, 2]$$
$$\beta = 1$$
$$dist(s, \mathcal{N}) = \min_{s' \in \mathcal{N}} |s - s'| = \begin{cases} 0 & , \ s \in [-2, 2] \\ |s| - 2 & , \ s \notin [-2, 2] \end{cases}$$

The performance functions are shown in Figure 5 and the cross hatched area represents the relaxed "forbidden" region

$$\mathcal{F}_{strict}(q) = \{(s, y) : s \in (-\infty, \infty), y < 10)\} \tag{35}$$
$$\mathcal{F}_{relaxed}(q) = \{(s, y) : s \in (-\infty, \infty), y < 10 - dist(s, \mathcal{N})\} \tag{36}$$
$$\mathcal{F}_{normal}(q) = \{(s, y) : s \in [-2, 2], y < 10\} \tag{37}$$

Observe that according to the radius of stability model based on these performance functions, the most robust, hence, optimal decision is q'''.

So, by inspection, decisions q''' and q'''' are inadmissible: Their graphs intrude into the forbidden region $\mathcal{F}$.

To decide which decision is optimal, we consider the worst values of $f(q', s)$ and $f(q', s)$ over the uncertainty space $(-\infty, \infty)$, and select the best worst case. This is shown in Figure 6.

The worst case of $f(q', s)$ is attained at $s = -7$ for which we have $f(q', -7) = 5$ and the worst case of $f(q'', s)$ is attained at at $s = 0$ for which we have $f(q', 0) = 0$. Hence, the best worst case is generated by q' and it is therefore the optimal decision in this case.

Observe that the worst case of $f(q'''', s)$ is attained at $s = 10$ and is equal to $f(q'''', 10) = 6$, which is better than the worst case of the optimal decision q'. However, as noted above, decision q'''' is inadmissible because it violates the "relaxed" global constraint (30).

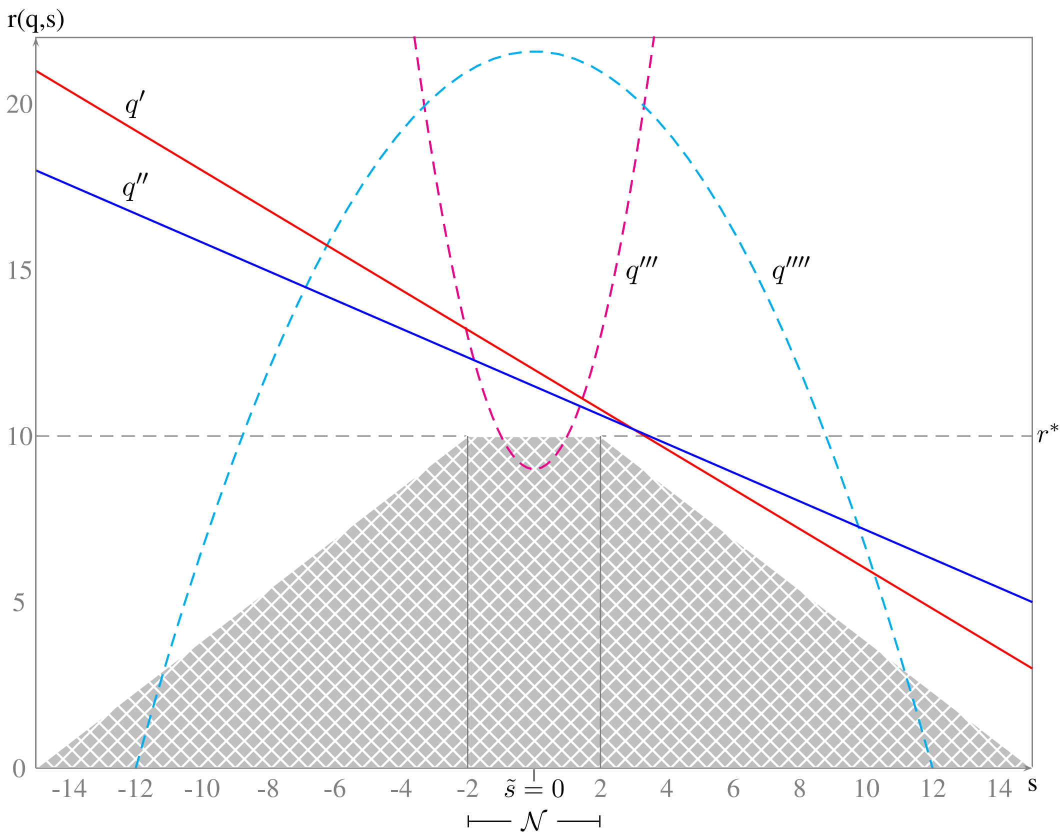

Figure 5: Forbidden region $\mathcal{F}_{relaxed}(q)$

8 INFO-GAP DECISION THEORY REVISITED

There are a number of points that deserve attention concerning info-gap decision theory's (mis)application of the radius of stability model in the analysis and management of severe uncertainty. It is also important to clarify certain issues, discussed in articles published in this journal, on the relation between info-gap's robustness model and Wald's maximin model.

8.1 Fooled by local robustness

First, it is important to point out that info-gap decision theory's misapplication of the radius of stability model is a manifestation of a systematic blurring of the lines — presumably due to a lack of awareness of the distinction — between *local* and *global* robustness. This seems to have instilled in proponents and users of this theory an unwarranted sense of *confidence* in the merits of info-gap's robustness analysis. For how else can one explain the prescription to use a radius of stability model to quantify, analyze and manage severe uncertainty that is characterized by poor estimates, likelihood-free models of uncertainty, and vast (even unbounded) uncertainty spaces?

Hence, it is important to be aware of the background of statements (such as those below) which describe the mode of operation of info-gap's robustness model and the results that it yields. Particularly, it is important to realize that these statements **have no global significance whatsoever.**

For instance, Yemshanov et al. [37, p. 262] maintain that info-gap's robustness analysis address the following question:

> How wrong can the data and underlying models be, while the outcome of the decision in question remains acceptable?

And they claim that [37, p. 266]:

> We used the info-gap robustness function as a decision tool to select the pest survey network that is most immune to uncertainties about the pest.

And Burgman et al. [38, p. 2] state that

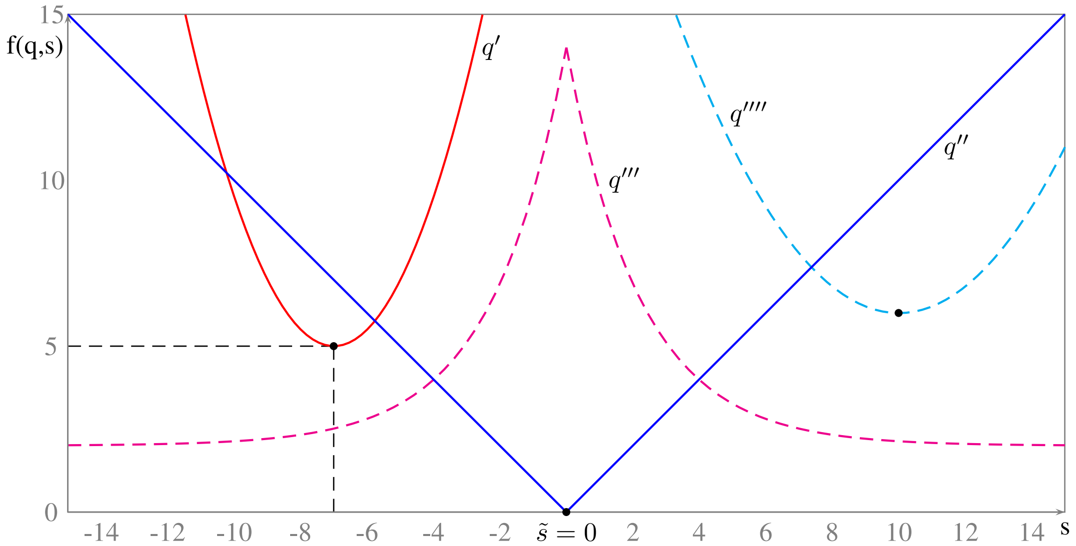

Figure 6: Objective functions and best worst case

> In general terms, info-gap theory finds decisions that achieve a minimally acceptable (satisfactory) outcome in the face of nonstatistical uncertainty, given a nominal estimate of the system.

Our point is that it is important to keep in mind that the results referred to are yielded by a *local* robustness analysis so that these statements should actually read as follows:

> We used the info-gap **local** robustness function as a decision tool to select the pest survey network that is most immune to uncertainties about the pest **in the neighborhood of the given estimate of the parameter of interest.**

and

> In general terms, info-gap theory finds decisions that achieve a minimally acceptable (satisfactory) outcome in the face of nonstatistical uncertainty **in the neighborhood of a given nominal estimate of the system.**

respectively.

This blurring of the lines between *local* and *global* robustness seems to explain a number of (erroneous) assertions about this theory in the info-gap literature. For instance, that this theory is particularly suitable for the treatment of *unbounded* uncertainty spaces, that info-gap's robustness model can be used to analyze and manage *unknown unknowns and Black Swans* [39] and that info-gap's robustness analysis explores the *entire* uncertainty space [39,40].

8.2 Place and role in the state of the art

Given the discussions in the preceding sections, it is clear that info-gap's robustness model ought to be viewed as an instance of two well-known and well-established models of robustness:

- Wald's Maximin model (circa 1940).
- Radius of stability model (circa 1960).

This means that info-gap decision theory can be described from these two related but distinct perspectives: the *stability theory* perspective and the *classical decision theory* perspective, keeping in mind, as we show above, that the radius of stability model itself is an instance of Wald's Maximin model.

One of the reasons that it is important to look at info-gap decision theory from these perspectives is that the issue here is not only "perspectives". We are dealing here with well-established areas of expertise with vast literatures and knowledge bases.

8.2.1 *Radius of stability perspective*

Viewed from this perspective, info-gap's robustness model constitutes a radius of stability model that is characterized by stability conditions of the form specified by (7), namely the stability condition is of the form $r^* \leq r(q, s)$.

Keeping this perspective in mind should immune users of this model to the *fooled by local robustness* syndrome. The picture below leaves no room for debate: the robustness analysis is *local,* hence it is unsuitable for the treatment of severe uncertainty.

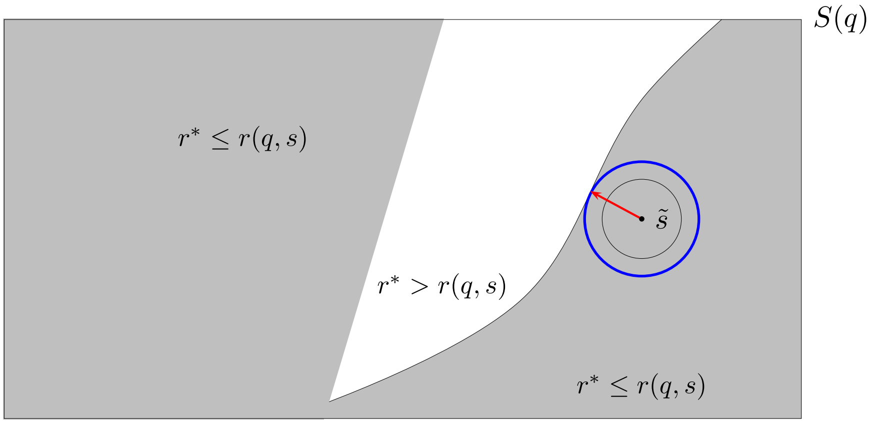

Figure 7: Info-gap's robustness model

It is important to call attention to this fact because, in the info-gap literature itself, there is no reference whatsoever to the connection between info-gap's robustness model and the radius of stability model.

8.2.2 *Wald's Maximin perspective*

This perspective combines the points of view of classical decision theory and robust optimization. Thus, conceptually, info-gap's robustness model is viewed as a *two players game:* the decision maker (DM) and *Nature*. The DM's objective is to inflate the largest possible balloon (ball) centered at the estimate $\tilde{s}$. Nature is antagonistic towards the decision maker so that it will puncture the balloon at the first opportunity, namely once the constraint $r^* \leq r(q, s)$ is violated at a state $s \in B(\rho, \tilde{s})$.

The end result is the following Maximin model:

$$\max_{\rho \geq 0} \min_{s \in B(\rho, \tilde{s})} f(\rho, s, q) \text{ where } f(\rho, s, q) := \begin{cases} \rho & , \ r^* \leq r(q, s) \\ -\infty & , \ r^* > r(q, s) \end{cases} \tag{38}$$

The penalty $-\infty$ is a reminder, to the DM, that the game is a *local worst-case game.* That is, for each value of ρ, the worst $s \in B(\rho, \tilde{s})$ determines whether the performance requirement is satisfied.

Interestingly, despite the formal proofs that info-gap's robustness model is an instance of Wald's Maximin model [10, 33–35], Ben-Haim [18] continues to maintain that info-gap's robustness model is not a maximin model. For example [18, p. 9],

> Info-gap theory is not a worst-case analysis. While there may be a worst case, one cannot know what it is and one should not base one's policy upon guesses of what it might be. Info-gap theory is related to robust-control and min-max methods, but nonetheless different from them. The strategy advocated here is *not* the amelioration of purportedly worst cases.

and [18, p. 10]

> The difference from min-max approaches is that we are able to select a policy without ever specifying how wrong the model actually is. Min-max and info-gap robust-satisficing strategies will sometime agree and sometime differ.

This misrepresentation of the relation between info-gap's robustness model and Wald's Maximin model, allegedly due to the fact that info-gap's robustness model does not require the existence and/or knowledge of a worst case, is also echoed in Yemshanov et al. [37, p. 274], Ben-Haim [41, p. 1,7] and Schwartz et al. [42].

It is important, therefore, to take note that info-gap's robustness analysis is clearly and unmistakably a (local) worst-case analysis. For a given value of ρ, the *worst* value of $s \in B(\rho, \tilde{s})$ determines whether ρ is admissible. That is, if the *worst* $s \in B(\rho, \tilde{s})$ satisfies the performance constraint, then ρ is admissible, otherwise it is not.

And the surprising thing is that the fact that info-gap's robustness analysis is a worst-case analysis is clearly indicated in Hemez et al. [43,44] and Hemez and Ben-Haim [45]. For example, the caption of Figure 7 in Hemez and Ben-Haim [45] is "Results of the worst-case info-gap robustness analysis".

The wider implication is that whereas Wald's generic Maximin model is sufficiently flexible to allow both a *local* and a *global* robustness analysis, info-gap's robustness analysis is *inherently local* because its robustness model is a *radius of stability model*.

8.2.3 *Discussion*

Setting off info-gap decision theory's declared aim: the pursuit of decisions that are robust to *severe* uncertainty, against these two perspectives, makes one thing abundantly clear. If there is anything that truly singles out info-gap decision theory as "new", different" etc., it is the fact that it is the only decision theory that advances the (untenable) idea that robustness against severe uncertainty can be obtained by a local robustness analysis in the neighborhood of a poor estimate.

As we saw above, this prescription for severe uncertainty is not only untenable (especially in cases where the uncertainty space is vast), it may well be deemed as representing "irresponssible" decision makers. Or, as Ben-Tal et al. [32] put it, robustness models (of severe uncertainty) that operate only on the "normal range" of values of a parameter — rather than on the entire range — represent a somewhat "irresponsible" decision-maker.

8.3 Case studies

To illustrate some of the consequences that result from this blurring of the lines between local and global robustness, we briefly examine concrete examples taken from five papers that were published in this journal.

8.3.1 *Managing the risk of uncertain responses*

In this paper, Lempert and Collins [46] report on "... a simple computer simulation model to compare several alternative frameworks for decision making under *deep* uncertainty ...".

In spite of the fact that some of the "alternative frameworks" referred to in this statement, are models of local robustness while others are models of global robustness, the comparison indiscriminately lumps these two types of models together to thereby obscure the fact that the results yielded by models of local robustness can be profoundly different from those yielded by models of global robustness.

In this vein, info-gap decision theory is cited in this study as an example of a method for robust decision-making under *deep* uncertainty possessing two important characteristics:

· Multiplicity of plausible futures.

· A satisficing criterion for desirable strategy.

It should also be pointed out that what the authors consider a distinctive feature of info-gap's robustness analysis, namely: "...trades a small amount of optimal performance for less sensitivity to broken assumptions..." is in fact a typical *Pareto tradeoff* that is performed after robustness had already been determined. In other words, this *Pareto tradeoff* is not an integral part of info-gap's definition of robustness, but an extra feature added on to it. Unsurprisingly, therefore, the authors are unaware that such a *Pareto tradeoff* can be easily incorporated in another method included in their study, namely the (global) method that is based on "satisficing over a wide range of futures" criterion.

Still, the main point is that info-gap's robustness model, which can yield only *local* robustness, is treated in this study as a model for the management of *deep* uncertainty. More on this article in Section 10.

8.3.2 *Evaluating critical uncertainty thresholds*

In this paper, Koch et al. [47] propose a methodology "...to assess the impact of uncertainties on the stability of pest risk maps as well as to identify geographic areas for which management decisions can be made confidently, regardless of uncertainty...".

Although info-gap decision theory was not used in the development of the proposed methodology, there is a statement about it in the article which requires pointing out. This statement attributes info-gap decision theory capabilities that it does not/cannot possibly have. Thus, on page 1238 we read that (emphasis is added):

> Another advance of the methodology presented here is the development of metrics for quantifying the impact of uncertainty in risk map products. The study illustrates how these metrics (e.g., S_{XY} and the shifts in risk classes) can be used to sketch out a horizon of parameter variation beyond which the output map loses its stability. However, the method may not be applicable in cases in which information for parameterizing the risk model is severely limited, as it requires at least approximate knowledge of the baseline parameter values. **We believe that such cases of severe uncertainty necessitate a different approach that does not rely on a probabilistic representation of the structure of uncertainty, such as the info-gap framework.**

As revealed in the follow-up to this article, namely in Yemshanov et al. [37], this misconception about info-gap decision theory's capabilities, which erroneously attributes it 'global' capabilities, had subsequently lead the authors to the use of "an info-gap framework" for the management of severe uncertainty.

8.3.3 *Robustness of risk maps and survey networks*

In this article, Yemshanov et al. [37] propose the use of info-gap decision theory to determine how much uncertainty in risk model assumptions can be tolerated before a risk map loses its value. As above, the uncertainty is severe in the sense that the uncertainty model is likelihood-free and it is impossible to specify exactly the values of the actual deviations from the nominal values.

Although the term "robustness" is rampant in the discussion, one would be hard pressed to appreciate that the robustness under consideration is local rather than global. It is important therefore to take note that the robustness obtained by the authors' analysis **does not** pertain to the severe uncertainty in the true values of the parameters, but rather to perturbations in the given nominal values of the parameters. As explained above, these are two radically different things.

It is interesting to note that although there is a reference in the paper to Wald's work (page 274), there is no indication that info-gap's robustness model is an instance of Wald's Maximin model.

8.3.4 *Robustness to uncertain correlations of estimates of the probability of a null event*

In this paper, Ben-Haim [41] uses info-gap decision theory to evaluate the robustness, to uncertain correlations, of estimates of the probability of a null event.

There is a short discussion in the paper on Wald's 1945 paper [23] but no clue to the fact that info-gap's robustness model is an instance of Wald's model.

But most interesting of all is the fact that the uncertainty model itself is a *one-dimensional* model. Namely, the parameter of interest is a numeric scalar (probability) and the uncertainty space is $\mathscr{U} = [-\alpha, 1-\alpha]$, where $\alpha \in [0,1]$ is a given numeric scalar.

This means, of course, that info-gap's robustness model is utterly superfluous, as all we need to do to determine the robustness of the system against perturbations in the nominal value $\tilde{u} = 0$, is to simply solve the following problem:

> Determine the range of values of $u \in \mathscr{U}$ that satisfy the constraint $|(1-\alpha-u)^{1/n} - p_e| \leq \delta$, where n is a given positive integer, and $p_e \in [0,1]$ and $\delta \in [0, p_e]$ are given numeric constant.

In such cases the critical values of the parameter (end points of the acceptable region of uncertainty surrounding the estimate) would be deduced directly from the performance requirement and would thus be thoroughly independent of the metric used to specify the radius of the balls (intervals) around the estimate.

Indeed, recalling the equivalence

$$|expression| \leq \delta \iff -\delta \leq expression \leq \delta \tag{39}$$

the following straightforward derivation yields the two end points of the range of acceptable values of u:

$$\left\{ \begin{array}{c} |(1-\alpha-u)^{1/n} - p_e| \leq \delta \\ -\delta \leq (1-\alpha-u)^{1/n} - p_e \leq \delta \\ p_e - \delta \leq (1-\alpha-u)^{1/n} \leq p_e + \delta \\ (p_e-\delta)^n \leq 1-\alpha-u \leq (p_e+\delta)^n \\ 1-\alpha-(p_e+\delta)^n \leq u \leq 1-\alpha-(p_e-\delta)^n \end{array} \right\} \tag{40}$$

Hence, the admissible range of u is $[u_l, u_u]$, where

$$u_l = 1-\alpha-(p_e+\delta)^n \ \ ; \ \ u_u = 1-\alpha-(p_e-\delta)^n \tag{41}$$

The length of this interval, namely $(p_e+\delta)^n - (p_e-\delta)^n$, can serve as the measure of the system's robustness. Alternatively, if the interval contains the nominal point $\tilde{u} = 0$, the distance to the two end points of this interval $\tilde{u} = 0$ can be scaled by different weights, if necessary/desirable, to determine the robustness (radius of stability) of the system.

The inference is therefore clear: not only is info-gap's robustness model totally uncalled for in this analysis, all that its incorporation in the robustness analysis does is to unnecessarily complicate it.

Incidentally, this is not the only case. For example, Halpern et al. (48) and Rout et al. (49) also use info-gap's robustness model to analyze a "one-dimensional" robustness model. In fact, in the case of (49), the problem is even simpler, as the performance requirement is simply this:

$$d + uE \leq C \tag{42}$$

where E and C are positive numeric scalars, $u \in \mathscr{U} = [0,1]$ and d positive integer representing the decision variable.

That info-gap's robustness model is utterly redundant in this case as well is immediately clear, because to determine the robustness of d against perturbation in the value of u with respect to this requirement, all we need to do is to determine the range of values of u that satisfy this requirement. By inspection, the critical value of u, as a function of d, is

$$u^*(d) := \frac{C-d}{E} \tag{43}$$

hence the range of admissible values of u is

$$R(d) := [0, u^*(d)] = \left[0, \frac{C-d}{E}\right] \tag{44}$$

observing that the size (length) of this interval is $(C-d)/E$.

In practice it may be useful/desirable to scale this length by a *weight* — whose value may depend on d — to obtain the robustness of decision d.

As above, all that the employment of an info-gap robustnessl model does here is to complicate an otherwise trivial robustness problem.

8.3.5 *Sensitivity analysis of Bayes nets*

In this paper, Burgman et al. (38) propose the use of info-gap decision theory to evaluate the sensitivity of decisions to possible large errors in the underlying probability estimates and utilities associated with Bayes nets. The uncertainty is severe in the sense that the uncertainty model is likelihood-free and the uncertainty in the probabilities and utilities is represented by intervals (around estimates) whose size is uncertain.

The authors thus contend that:

> Info-gap decision analysis can be used in this context to maximize the robustness of a given model for a specified performance threshold, rather than to maximize expected performance.

But it is important to take note that, in the context that the authors discuss, the robustness yielded by their analysis **is not** (global) robustness against the *severe* uncertainty in the probabilities and utilities. Rather, it is (local) robustness against perturbations in the nominal values of the given estimates.

Indeed, the fact that the stated objective is to evaluate the robustness of a given decision, rather than finding the most robust decision, raises an even more fundamental question. Isn't it more "natural" in this case to use a proper *sensitivity analysis* model (e.g. radius of stability models already available for this particular purpose) rather than a *decision-making* model?

8.3.6 *Remark*

The fact that four out of the five case studies, as well as the articles by Halpern et al. [48] and Rout et al. [49], are in the areas of *applied ecology*, *conservation biology* and *environmental management,* is not accidental. Indeed, as indicated by the reference lists in these papers, info-gap decision theory seems to have gained quite a following in these disciplines. So much so that in Moilanen et al. [50, p. 123] we read:

> In summary, we recommend Info-Gap uncertainty analysis as a standard practice in computational reserve planning. The need for robust reserve plans may change the way biological data are interpreted. It also may change the way reserve selection results are evaluated, interpreted and communicated. Information-gap decision theory provides a standardized methodological framework in which implementing reserve selection uncertainty analyses is relatively straightforward.

Reviews of this and other articles dealing with applications of info-gap decision theory in these and other disciplines can be found at *http://info-gap.moshe-online.com/reviews.html*.

9 WHERE IS THE LIMIT?

Given that the *radius of stability* model (read, info-gap's robustness model) had been adopted as a medium for analysis and management of severe uncertainty in areas ranging from applied ecology, conservation biology, bio-security, economics, finance, medicine, risk analysis, law, to social behavior; the question that immediately springs to mind is this: *Are there any limits to what one can do with this model of* **local** *robustness/stability*?

Apparently, the answer to this intriguing questions is "no". There are no limits. The only limit is in one's imagination.

The point is that, once the distinction between *local* and *global* robustness is blurred, or ignored, or not appreciated, there is no limit to the potential (mis)application of the *radius of stability* model. The trouble of course is that analysts who are not well-versed in decision theory, optimization theory etc., may not be in a position to appreciate that this amounts to a misapplication of this model. Four example should suffice to illustrate this point.

9.1 Example

Consider the opening statement in the *Preface* to *Info-Gap Economics: an Operational Introduction* [18, p. x]:

> The management of surprises is central to the "economic problem", and info-gap theory is a response to this challenge.

Granted, it is safe to assume that many would agree with the proposition that the management of severe uncertainty should entail accounting for surprises, rare events, shocks and possible catastrophes. It is equally safe to assume that many would accept that models aiming to faithfully depict situations of severe uncertainty should seek to determine/rank decisions according to their *robustness*.

But, the question is: how can a theory that is based on a model of *local* robustness possibly be able to offer a "response to this challenge"?

The answer of course is that given that info-gap theory's robustness model is a radius of stability model, it cannot possibly meet the challenges presented by severe uncertainty, let alone surprises, extreme events, shocks, catastrophes, and so on.

However, because *info-gap decision theory* is nevertheless portrayed as a theory for the management of severe uncertainty, unwary reader are given the wrong impression about this theory's capabilities, its scope, and its applicability.

It is perhaps instructive to sum up this point using a phrase taken from a statement in the *Black Swan*. The latest edition of the book contains a new section entitled *Robustness and Fragility* where the opening statement reads as follows [2, p. 310]:

> ROBUSTNESS AND FRAGILITY
>
> Upon the completion of *The Black Swan,* I spent some time meditating on the items I raised in Chapter 14 on the fragility of some systems with large concentration and illusions of stability — which had left me convinced that the banking system was the mother of all accidents waiting to happen.

The phrase that is of interest to our discussion is: **illusions of stability**. Without attributing to Taleb our take on this phrase, our point is that a blurring of the lines between *local* and *global* robustness is a recipe for adopting the "wrong" robustness model for the management of severe uncertainty, hence for entertaining **illusions of stability** about its results.

9.2 Example

In the paper entitled *Allocating monitoring effort in the face of unknown unknowns* by Wintle et al. [39, p. 8] we read the following:

> The third type of model application would involve a formal uncertainty analysis that explores the full space of monitoring investment options and parameter uncertainties to identify the most robust monitoring investment (*sensu* Wald 1945, Ben-Haim 2006).

Here the *radius of stability* model deployed by info-gap decision theory is attributed a capability that it cannot possibly possess as a *radius of stability* model, namely the capability to explore "the full [uncertainty] space". So, the failure to appreciate that info-gap's robustness model is a *radius of stability* model, coupled with the failure to appreciate the inherent *local* orientation of this model, hence the implications of this fact, lead the authors to erroneously conclude that this model yields "the most robust monitoring investment" in the face of (*severe*) "parameter uncertainties".

9.3 Example

A common misconception arising from the failure to appreciate the *local* bent of the radius of stability model is that this model determines the **widest range** of an uncertain parameter values for which a satisfactory outcome is attained. For example, in Duncan et al. [51, p. 48] we read:

> Besides uncertainty modelling, IGDT is unique because of its decision principles. Using IGDT, a decision maker confronts uncertainty by maximising robustness to it, seeking reasonably satisfactory performance over the widest range of unknown uncertainty, i.e., the largest horizon of uncertainty, α.

Similarly, in a recent paper in *The Journal for the Theory of Social Behaviour*, Schwartz et al. [42] propose *Robust Satisficing* as a new normative standard for rational decision-making in the face of *radical* uncertainty. In the abstract we read:

> Instead of seeking to maximize the expected value, or utility, of a decision outcome, robust satisficing aims to maximize the robustness to uncertainty of a *satisfactory* outcome. That is, robust satisficing asks, "what is a 'good enough' outcome," and then seeks the option that will produce such an outcome under the widest set of circumstances.

Of interest to this discussion is the phrase: **the widest set of circumstances**. The assertion that "robust satisficing" (read info-gap's robustness model) "seeks an option . . . under the widest set of circumstances" ascribes this model a capability that it clearly does not/cannot have as a *radius of stability* model. It thus gives the wrong impression that the proposed robustness model is a model of *global* robustness that explores the entire uncertainty space to determine the set of all satisfactory outcomes.

9.4 Example

Perhaps more than all the preceding examples, the summary shown in Figure 8 is indicative of the blurring of the distinction between *local* and *global* robustness in the info-gap literature.

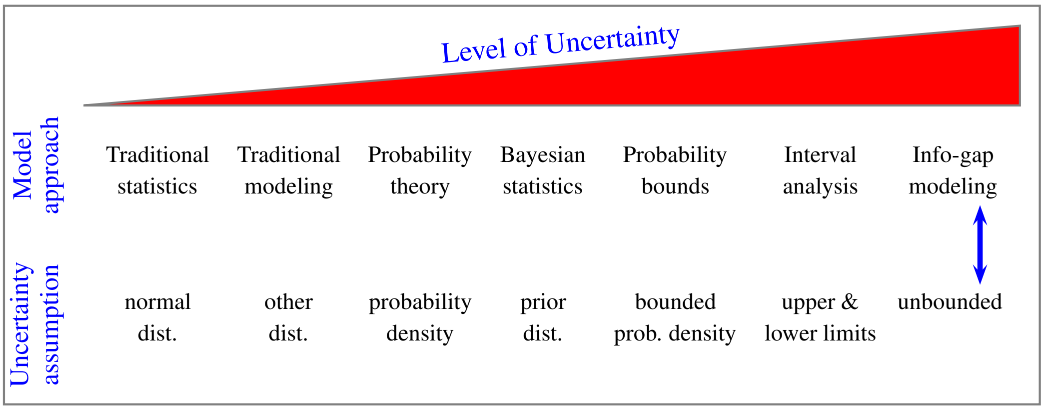

Figure 8: Treatment of various levels of uncertainty

This summary is a reproduction of the assessment given by Halpern et al. [(48)] of the domain of applicability of various methods and approaches in relation to the severity of the uncertainty under consideration.

It presents a likelihood-free model of LOCAL robustness that seeks decisions that are robust against small perturbations in the nominal value of a parameter of interest (namely info-gap's robustness model) as a tool for the modeling, analysis and management of severe uncertainty that is characterized by an UNBOUNDED UNCERTAINTY SPACE!

9.5 Safeguards

Having said all that, the question obviously arising is whether it is possible to safeguard against the adverse effects of a blurring of *local* and *global* robustness. And the most obvious answer to this is that authors should take care to be explicit about the type of robustness they are considering.

For instance, consider the following unambiguously clear description of the robustness under investigation [(52, p. 2067)]:

> This paper develops a general framework for conducting local robustness analysis. By local robustness, we refer to the calculation of control solutions that are optimal against the least favorable model among models close to an initial baseline.

In contrast, consider this loosely formulated, in fact misleading depiction of the robustness (purportedly) sought [(42, p. 22)]:

> A business operating with an eye toward robust satisficing asks not, "How can we maximize return on investment in the coming year?" It asks, instead, "What kind of return do we want in the coming year, say, in order to compare favorably with the competition? And what strategy will get us that return under the widest array of circumstances?"

Note that here the robustness model that supposedly answers these questions is a *radius of stability model* (read: info-gap's robustness model). However, as there is no reference in the entire discussion to the *local* nature of the model, much less to the difference between *local* and *global* robustness, unwary readers may not be in a position to appreciate that the proposed strategy **is in principle unable to** "get us that return under the widest array of circumstances".

Finally, not only is it important that authors take care to be clear on the type of robustness models they are discussing (local or global), it is equally important that referees of journals be vigilant about this point.

10 ROBUST OPTIMIZATION REVISITED

As a final note it should be pointed out that the absence of a distinction between *local* and *global* robustness, in publications advocating the use of info-gap decision theory in the management of severe (radical, deep) uncertainty, more precisely decision in the face of severe uncertainty, is part of a deeper problem. This is the total oblivion exhibited in these publications to an area of expertise that has an immediate relevance to info-gap decision theory's main concerns, namely the field of *robust optimization*.

The reason that this fact must be brought again here is not only because, as indicated in Section 6, info-gap's robustness model is in fact a simple radius of stability model, hence, a rather simple, *local robust optimization* model. But also because the body of knowledge that has accumulated over the past forty years in the study of robust optimization, bears directly on the methodology proposed by info-gap decision theory. Thus, the issues, models, algorithms, results etc. that are discussed in the rich literature on *robust optimization* either shed a more accurate light on the management of severe uncertainty, or preempt the models proposed in info-gap publications, or invalidate many of the claims made in these publications, and so on and so forth.

The same, needless to say, ought to be pointed out about the absence from the info-gap literature of all reference to the fact that info-gap's robustness model is a simple *radius of stability model* and to the vast literature on models of stability. For details on the importance of stability models (local and global) in the development of modern of robust optimization [14, pp. xvi-xvii].

To attempt to explain why the info-gap literature is completely oblivious to the existence of the field of *robust optimization* one would have to engage in speculation. Still, it is possible to point to two facts that might provide some sort of explanation to this state-of-affairs.

First, as indicated in Section 5, from its inception info-gap decision theory has been proclaimed a unique theory that is radically different from all current theories for decision under uncertainty. One imagines therefore that the link to *robust optimization* has not been identified because no serious attempt has been made to identify links between info-gap decision theory and other established theories, methods, paradigm etc., dealing with robust decision in the face of severe uncertainty.

Second, from its inception, info-gap decision theory has been portrayed, indeed hailed, as a theory pursuing *robust satisficing* rather than *robust optimizing*. The rationale advanced in the info-gap literature for this position has been that in the face of *severe* uncertainty, *satisficing* (the satisfying of requirements) has an advantage over *optimizing* (the maximization of an objective function). Incidentally, this rationale is also behind the proposition in Schwartz et al. [42] cited above, to base rational decision-making in the face of uncertainty on a "new" paradigm rooted in *robust satisficing* (read: info-gap decision theory), rather than on utility maximization.

Indeed, this distinction between *robust satisficing* and *robust optimizing* seems to have instilled the idea that because the info-gap methodology aims to accomplish different goals than those sought by (presumably) "standard" optimization methods, it also does different things than those performed by a (presumably) "standard" robust optimization methods.

Although this fact provides some sort of explanation for info-gap's isolation from *robust optimization,* it should be noted that this isolation is of course unjustified. To begin with, the rational that it seems to be based on, namely the distinction between *robust satisficing* and *robust optimizing,* is without any merit, because, *robust satisficing* is a specific case of *robust optimizing,* the obvious reason being that *satisficing* is an integral part of (constrained) *optimizing*. In other words, any satisficing problem can be formulated as an equivalent optimization problem, and any robust satisficing problem can be formulated as an equivalent robust optimization problem.

But more than this, formally, info-gap's robustness model (3) and info-gap's decision model (6) are in fact *optimization* models, where the *robustness* of a decision is defined by an optimization problem that is subject to a robustness *constraint*. This means of course that they these models are *robust optimization* models. Indeed, from the standpoint of robust optimization, these models are *simple robustness models* in the sense that: (a) the objective function is *equal to one of the decision variables*, namely ρ, and (b) they stipulate *only one* robustness constraint.

And to sum it all up, info-gap's decision theory's isolation from *robust optimization* is not only without any foundation, it is in fact harmful. Info-gap's "isolationist" stance gives users of this methodology a thoroughly distorted view on what the quest for robustness under severe uncertainty involves, and about the progress that has been made, in the last decades, in the study of this topic.

11 CONCLUSIONS

Although the term *robustness* has a familiar ring to it, it can have different meanings in different contexts. It is important therefore to be clear on the meaning or definition that this term is intended to have in a specific investigation.

Thus, phrases such as: *"robust satisficing"* and *"robust optimizing"*, and statements such as: *"this theory selects the most robust decision"* and *"this decision is more robust than that decision "* must be carefully, hence unambiguously defined. That is, their interpretation must be consistent with the formal definition of the robustness model in which context they are used. Meaning that a statement such as: "get us that return under the widest array of circumstances" must be shown to be consistent with the formal definition of the model's mode of operation and its capabilities.

In this vein, it is important to take note that the *radius of stability* model is a model of *local* robustness designed for the quantification, modeling, analysis and management of *small perturbations* in the nominal value of a parameter.

As such, it is unsuitable for the analysis and management of situations subject to severe uncertainty in the true value of the parameter of interest, especially in cases where the severity of the uncertainty is manifested in a poor estimate, vast region of uncertainty and a likelihood-free quantification of uncertainty. The more severe the uncertainty, the more unsuitable this model is for this purpose.

It is important to avoid interpreting the *local robustness* measured by the radius of stability model as a measure of *global robustness*, as this may induce an unwarranted confidence in decisions that are only locally robust. And it may induce rejection of decisions that are globally robust but locally fragile.

Experience with info-gap decision theory over the past ten years has shown that this danger is real. I therefore refer info-gap scholars to the following quote:

> **Author Summary**
>
> Robustness is an intrinsic property of many biological systems. To quantify the robustness of a model that represents such a system, two approaches exist: global methods assess the volume in parameter space that is compliant with the proper functioning of the system; and local methods, in contrast, study the model for a given parameter set and determine its robustness. Local methods are fundamentally biased due to the a priori choice of a particular parameter set. Our 'glocal' analysis combines the two complementary approaches and provides an objective measure of robustness. We apply this method to two prominent, recent models of the cyanobacterial circadian oscillator. Our results allow discriminating the two models based on this analysis: both global and local measures of robustness favor one of the two models. The 'glocal' method also identifies key factors that influence robustness. For instance, we find that in both models the most fragile reactions are the ones that affect the concentration of the feedback component.
>
> Hafner et al. (2009, p. 1) [53]

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